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Modelling Impact of Assumed Vaccine of Covid-19 in Kenya

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Abstract: **This work investigates the optimal control of the Covid-19 model. The model consists of susceptible, exposed, infected, quarantine, hospitalized, complicated, recovered, and death compartments. We present different components of the model and their interactions. We also give Numerical simulations to illustrate and compare the obtained results.**

Keywords: **Covid-19, Mathematical model, optimal control, SEIR, Stability Analysis, vaccination.**

I. INTRODUCTION

The Covid-19 pandemic (coronavirus pandemic) is a persistent disease that emerged in Wuhan, China, at the end of December 2019 [5]. It spread rapidly outside the Chinese borders all over the world. More than 15,785,641 confirmed cases worldwide and around 640,016 deaths up to July 26, 2020 [10]. On March 11, 2020, The World Health Organization (WHO) recognized the spread of Covid-19 as a pandemic. In Kenya [11], the first imported Covid-19 case was confirmed on March 13, 2020; it involved a 27-year-old Kenyan woman who traveled from the US via London. By 1^{st} April 2020, total Covid-19 cases rose to 81. The number of confirmed cases increased gradually while the government was putting measures to mitigate the spread.

The spread of the disease suddenly changed, and many infections were reported between June and September. This sudden increase in the number of new cases suggested that the existing forecasting tools are inadequate in the presence of a possible vaccine and second wave. That is why in this paper, we propose a mathematical model as a tool to help simulate and analyze the effect of the imperfect-assumed vaccine.

Mathematical modeling of infectious disease performs a key role in efforts that focus on forecasting, evaluating, controlling, and planning effective control strategies. For the Covid-19 pandemic, several models have been developed. Lin *et al.*[7], extended an SEIR model that captures the course of the Covid-19 outbreak and sheds light on understanding the trends of the outbreak. Zhang *et al.* [12] estimate the reproductive number R_0 of the novel virus in the early stage of the outbreak and make a prediction of daily new cases on the Diamond Princess cruise ship, the distribution of R_0 was about 2.28. Giordano *et al.* [3] developed a mathematical model based on susceptible, infected, diagnosed, recognized, threatened, healed, extinct. The results provided policymakers with a tool to assess the consequences of possible strategies, including lock-down and social distancing and testing and contact tracing. In this study, a novel mathematical model with nine compartments: susceptible, vaccinated, exposed, also known as (pre-asymptomatic), infected with mild symptoms (asymptomatic), Quarantined in a home with mild symptoms, Quarantined in hospitalized with complication, Quarantined in the hospital with breathing assistance, treatable and mortality due to disease, is formulated to describe the spread of Covid-19 in the population. Currently, many countries have developed vaccines whose efficacy has not been established. The focus of this paper is the forecasting of Covid-19 transmission in the presence of an assumed vaccine. The results of this paper will inform health authorities and relevant stakeholders on the optimum control strategies for the disease in Kenya.

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II. MODEL

S are the susceptible population, the V vaccinated, E exposed (pre-asymptomatic), I infected with mild symptoms (asymptomatic), Q Quarantined in a home with mild symptoms, H Quarantined in the hospital with complication, C Quarantined in the hospital with breathing assistance, R treatable and D mortality due to disease.

$$\begin{aligned}
\frac{dS}{dt} &= -\beta_1\frac{SE}{N} - \beta_2\frac{SI}{N} - \lambda_1 S \\
\frac{dV}{dt} &= \lambda_1 S - \beta_3\frac{VE}{N} - \beta_4\frac{VI}{N} - \lambda_2 V \\
\frac{dE}{dt} &= \beta_1\frac{SE}{N} + \beta_2\frac{SI}{N} + \beta_3\frac{VE}{N} + \beta_4\frac{VI}{N} - \theta E \\
\frac{dI}{dt} &= \theta E - (\gamma_1 + \gamma_2 + \gamma_3)I \\
\frac{dQ}{dt} &= \gamma_1 I - (\sigma_1 + \delta_1)Q \\
\frac{dH}{dt} &= \gamma_2 I + \sigma_1 Q - (\sigma_2 + \delta_2)H \\
\frac{dC}{dt} &= \gamma_3 I + \sigma_2 H - (\mu + \delta_3)C \\
\frac{dR}{dt} &= \delta_1 Q + \delta_2 H + \delta_3 C + \lambda_2 V \\
\frac{dD}{dt} &= \mu C
\end{aligned} \tag{1}$$

We assume in this model that the vaccination does not give perfect protection again the COVID-19. Therefore, could be the rate of been vaccinated λ_1, and λ_2 is the rate of vaccinated people who have gained immunity and become recovered. . We assume that each infectious sub-population (E and I) infected the healthy population with different density, the per capita density of infection is given by β_i. Our main assumption of the imperfection of the vaccination is based on the fact that some vaccinated people might get only partial protect and they can be infected if there are exposed to multiple infections. The imperfection can be due to the mutation of the virus. When people get vaccinated, they lean to lower their guards and they take fewer protection measures again the virus. $\frac{1}{\theta}$ is the incubation period of the infection. Some people show mild symptoms with per capita rate γ_1 and can stay at home with treatment, others develop hard symptoms and must be monitored in the hospital with per capita rate γ_2 and other critical conditions that require penetrating breathing with per capita rate γ_3. The parameter δ_i, with $i = 1,2,3$ represents the recovery rates of quarantined, hospitalize and critical (respectively). $\frac{1}{\sigma_i}$, with $i = 1,2$ duration of the quarantine period and the hospitalized period. Finally, μ is the death rate.

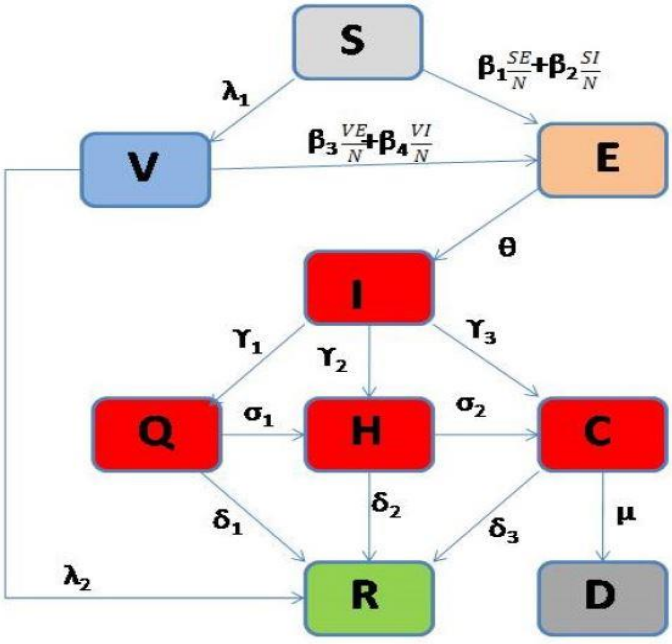

Figure 1: Flow chart of the model

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The system (1) that satisfies a given initial condition $(S(0),\ V(0),\ E(0),\ I(0)\ Q(0),\ H(0),\ C(0),\ R(0),\ D(0))$ has a unique solution.

Proof. We rewrite system (1) in the following form

$$\dot{X} = AX + F(X) \tag{2}$$

where $X(t) = [S(t),\ V(t),\ E(t),\ I(t),\ Q(t),\ H(t),\ C(t),\ R(t),\ D]^{\top}$,

$$A = \begin{bmatrix} -\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_1 & -\lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\theta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta & -\gamma_1-\gamma_2-\gamma_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_1 & -\sigma_1-\delta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_2 & \sigma_1 & -\sigma_2-\delta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_3 & 0 & \sigma_2 & -\mu-\delta_3 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & \delta_1 & \delta_2 & \delta_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 \end{bmatrix},$$

$$F(X) = \begin{bmatrix} -\beta_1 \frac{SE}{N} - \beta_2 \frac{SI}{N} \\ -\beta_3 \frac{VE}{N} - \beta_4 \frac{VI}{N} \\ \beta_1 \frac{SE}{N} + \beta_2 \frac{SI}{N} + \beta_3 \frac{VE}{N} + \beta_4 \frac{VI}{N} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Equation (2) is a non-linear system with a bounded coefficient.

we set

$$D(X) = AX + F(X)$$

We have

$$\begin{aligned} |F(X_1) - F(X_2)| &\leq \tfrac{2\beta_1}{N}|S_2E_2 - S_1E_1| + \tfrac{2\beta_2}{N}|S_2I_2 - S_1I_1| + \tfrac{2\beta_3}{N}|V_2E_2 - V_1E_1| + \tfrac{2\beta_4}{N}|V_2I_2 - V_1I_1| \\ &\leq \tfrac{2\beta_1}{N}|S_1(E_2 - E_1) + E_2(S_2 - S_1)| + \tfrac{2\beta_2}{N}|S_1(I_2 - I_1) + I_2(S_2 - S_1)| + \\ &\quad \tfrac{2\beta_3}{N}|V_1(E_2 - E_1) + E_2(V_2 - V_1)| + \tfrac{2\beta_4}{N}|V_1(I_2 - I_1) + I_2(V_2 - V_1)| \\ &\leq 2\beta_1(|E_2 - E_1| + |S_2 - S_1|) + 2\beta_2(|I_2 - I_1| + |S_2 - S_1|) + \\ &\quad 2\beta_3(|E_2 - E_1| + |V_2 - V_1|) + 2\beta_4(|I_2 - I_1| + |V_2 - V_1|) \\ &\leq 2(\beta_1 + \beta_3)|E_2 - E_1| + 2(\beta_1 + \beta_2)|S_2 - S_1| + 2(\beta_2 + \beta_4)|I_2 - I_1| + 2(\beta_3 + \beta_4)|V_2 - V_1| \\ &\leq 4M(|S_2 - S_1| + |V_2 - V_1| + |E_2 - E_1| + |I_2 - I_1|), \quad where \quad M = \max\{\beta_1, \beta_2, \beta_3, \beta_4\} \\ &\leq 4M \parallel X_2 - X_1 \parallel \end{aligned}$$

then, we get $|D(X_1) - D(X_2)| \leq \overline{M} \parallel X_1 - X_2 \parallel$, where $\overline{M} = \max\{M, \parallel A \parallel\} < \infty$. Thus, it follows that the function D is uniformly Lipschitz continuous. From the definition of the control $u(t)$ and the restriction on $S(t)$, $V(t)$, $E(t)$, $I(t)$, $Q(t)$, $H(t)$, $C(t)$, $R(t)$, and $D(t) > 0$, we see that a solution of the system (2) exists [1].

If $S(0) \geq 0,\ V(0) \geq 0,\ E(0) \geq 0,\ I(0) \geq 0\ Q(0) \geq 0,\ H(0) \geq 0,\ C(0) \geq 0,\ R(0) \geq 0$ and $D(0) \geq 0$, the solutions $S(t), V(t), E(t), I(t)\ Q(t), H(t), C(t), R(t), D(t)$ of system (1) are positive for all $t \geq 0$.

Proof. It follows from the first equation of system (1) that

$$\frac{dS(t)}{dt} = -\beta_1 \frac{S(t)E(t)}{N} - \beta_2 \frac{S(t)I(t)}{N} - \lambda_1 S \tag{3}$$

$$= -(\beta_1 \frac{E(t)}{N} + \beta_2 \frac{I(t)}{N} + \lambda_1)S(t) \tag{4}$$

$$= -F(t)S(t) \tag{5}$$

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where $F(t) = \beta_1 \frac{E(t)}{N} + \beta_2 \frac{I(t)}{N} - \lambda_1$. The both sides in last equality are multiplied by $exp(\int_0^t F(s)ds)$, we obtain $\frac{dS}{dt} exp(\int_0^t F(s)ds) + F(t)S(t)exp(\int_0^t F(s)ds) = 0$ then $\frac{d}{dt}()S(t)exp(\int_0^t F(s)ds) = 0$. Integrating this inequality from 0 to t gives $S(t) = S(0)exp(-\int_0^t F(s)ds) \geq 0$. Similarly, we prove that $V(t) \geq 0$, $E(t) \geq 0$, $I(t) \geq 0$, $Q(t) \geq 0$, $H(t) \geq 0$, $C(t) \geq 0$, $R(t) \geq 0$, $D(t) \geq 0$.

III. THE BASIC REPRODUCTION NUMBER

The system (1) has always the disease free equilibrium point $E_0 = (N, 0,0,0,0,0,0,0)$. The basic reproduction number R_0 is the average number of people that one person with the Covid-19 is likely to infect in a susceptible population and it is calculated using the approach of next generation matrix [9]. Therefore, the disease compartments are E and I.

$$F = \begin{pmatrix} \beta_1 \frac{S^*E^*}{N} + \beta_2 \frac{S^*I^*}{N} + \beta_3 \frac{V^*E^*}{N} + \beta_4 \frac{V^*I^*}{N} \\ 0 \end{pmatrix} V = V^- - V^+ = \begin{pmatrix} \theta E \\ (\gamma_1 + \gamma_2 + \gamma_3)I^* - \theta E^* \end{pmatrix} \quad (6)$$

The matrices F and V are:

$$F = \frac{\partial F(E_0)}{\partial X_j} = \begin{pmatrix} \beta_1 & \beta_2 \\ 0 & 0 \end{pmatrix} \quad (7)$$

$$V = \frac{\partial V(E_0)}{\partial X_j} = \begin{pmatrix} \theta & 0 \\ -\theta & (\gamma_1 + \gamma_2 + \gamma_3) \end{pmatrix} \quad (8)$$

$$V^{-1} = \begin{pmatrix} \frac{1}{\theta} & 0 \\ \frac{1}{(\gamma_1+\gamma_2+\gamma_3)} & \frac{1}{(\gamma_1+\gamma_2+\gamma_3)} \end{pmatrix}$$

Therefore

$$R_0 = \rho(FV^{-1}) = \frac{\beta_1}{\theta} + \frac{\beta_2}{(\gamma_1+\gamma_2+\gamma_3)} \quad (9)$$

A. *Local Stability Analysis at Disease-Free Equilibrium (DFE) E_0*

The local stability of the DFE is analyzed as follows.

The Jacobian matrix of the ODE system (1) at $E_0 = (N, 0,0,0,0,0,0)$ is

$$J^*(E_0) = \begin{pmatrix} -\lambda_1 & 0 & -\beta_1 & -\beta_2 & 0 & 0 & 0 \\ \lambda_1 & -\lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 - \theta & \beta_2 & 0 & 0 & 0 \\ 0 & 0 & \theta & -(\gamma_1 + \gamma_2 + \gamma_3) & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_1 & -(\sigma_1 + \delta_1) & 0 & 0 \\ 0 & 0 & 0 & \gamma_2 & \sigma_1 & -(\sigma_2 + \delta_2) & 0 \\ 0 & 0 & 0 & \gamma_3 & 0 & \sigma_2 & -(\mu + \delta_3) \end{pmatrix}. \quad (10)$$

Thus, the eigenvalues of the Jacobian matrix $J^*(E_0)$ are the roots of the following characteristic equation

$$(-\lambda_1 - \lambda)(-\lambda_2 - \lambda)(-\sigma_1 - \delta_1 - \lambda)(-\sigma_2 - \delta_2 - \lambda)(-\mu - \delta_3 - \lambda)(\lambda^2 + a_1\lambda + a_0) \quad (11)$$

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Where

$$\begin{aligned} a_0 &= (\theta-\delta_1)(\gamma_1+\gamma_2+\gamma_3)-\theta\beta_2 \\ a_1 &= (\theta-\delta_1)+(\gamma_1+\gamma_2+\gamma_3) \end{aligned} \quad (12)$$

To calculate the roots of $(\lambda^2+a_1\lambda+a_0)$, let $\alpha_1=\beta_1-\theta$ and $\gamma=(\gamma_1+\gamma_2+\gamma_3)$ to obtain

$$\lambda_1=\frac{\beta_1-\theta-(\gamma_1+\gamma_2+\gamma_3)-\sqrt{(\beta_1-\theta+\gamma_1+\gamma_2+\gamma_3)^2+4\theta\beta_2}}{2}=\frac{\alpha_1-\gamma-\sqrt{(\alpha_1-\gamma)^2+4\theta\beta_2}}{2} \quad (13)$$

$$\lambda_2=\frac{\beta_1-\theta-(\gamma_1+\gamma_2+\gamma_3)+\sqrt{(\beta_1-\theta+\gamma_1+\gamma_2+\gamma_3)^2+4\theta\beta_2}}{2}=\frac{\alpha_1-\gamma+\sqrt{(\alpha_1-\gamma)^2+4\theta\beta_2}}{2} \quad (14)$$

It is clear that if $\alpha_1-\gamma<0$ then $\lambda_1<0$. In the case to prove that $\lambda_2<0$, we write the equation

$$1-R_0=\frac{-\alpha_1\gamma-\theta\beta_2}{\gamma\theta}>0 \quad (15)$$

From the previous equation if $R_0<1$ then

$$\begin{aligned} & \alpha_1\gamma+\theta\beta_2<0 \\ \Rightarrow\ & (\alpha_1+\gamma)^2+4\theta\beta_2<(\gamma-\alpha_1)^2 \\ \Rightarrow\ & \sqrt{(\alpha_1-\gamma)^2+4\theta\beta_2}<\gamma-\alpha_1 \\ \Rightarrow\ & \alpha_1-\gamma+\sqrt{(\alpha_1-\gamma)^2+4\theta\beta_2}<0 \\ \Rightarrow\ & \lambda_2<0 \end{aligned}$$

In conclusion if $(\alpha_1-\gamma)<0$ and $R_0<1$ then the roots of $(\lambda^2+a_1\lambda+a_0)$ are negative. Under these conditions all the roots of the characteristic equation (11) are negative. Thus we have just proved the following result: The disease free equilibrium E_0 of the system (1) is locally asymptotically stable if $R_0<1$ and unstable otherwise.

B. Global Stability Analysis at Disease-Free Equilibrium

In this section, we analyze the global asymptomatic stability of the disease-free equilibrium states. using the same approach [6] and applying the result introduced in paper [2]

Consider the model written in a vector form as follows:

$$\frac{dX_1}{dt}=F(X_1,X_2) \quad (16)$$

$$\frac{dX_2}{dt}=G(X_1,X_2),\quad G(X_1,0)=0 \quad (17)$$

Where X_1 denotes the uninfected compartments, X_2 denotes the infected compartments and $X_0=(X_1^*,0)$ represents the disease free equilibrium of the system.

Assume also the following two conditions:

1. $\frac{dX_1}{dt}=F(X_1,0)$, X_1^* is globally asymptotically stable.
2. $G(X_1,X_2)=AX_2-G^*(X_1,X_2)$, $G^*(X_1,X_2)\geq 0$ for all $(X_1,X_2)\in\Omega$.

where Ω is the region of the model, and the Jacobian $A=\frac{\partial G}{\partial X_2}(X^*,0)$ is M-matrix.

Then $X_0=(X^*,0)$ is globally asymptotically stable which provides $R_0<1$.

Theorem 1 *The disease free equilibrium $X_0=(X_1^*,0)$ of the model(1) is globally asymptotic stable if $R_0<1$.*

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Proof: From the system (1), $X_1 = (S, V)$, $X_2 = (E, I)$ and $X_1^* = (N, 0)$.

$$\frac{dX_1}{dt} = F(X_1, 0) = \begin{pmatrix} \lambda_1 S \\ \lambda_1 S - \lambda_2 V \end{pmatrix} \tag{18}$$

For the equilibrium point, $X_1^* = (N, 0)$ the system can be reduced to

$$\frac{dS}{dt} = -\beta_1 \frac{SE}{N} - \beta_2 \frac{SI}{N} - \lambda_1 S \tag{19}$$

$$\frac{dV}{dt} = \lambda_1 S - \beta_3 \frac{VE}{N} - \beta_4 \frac{VI}{N} - \lambda_2 V \tag{20}$$

Therefore, the characteristic equation of (19) is given by

$$\lambda^2 + (\lambda_1 + \lambda_2)\lambda + \lambda_1 \lambda_2 \tag{21}$$

There are two negative The roots of the equation (21) are $-2\lambda_1$ and $-2\lambda_2$.

Therefore, $X_1^* = (N, 0)$ is always globally asymptotically stable.

$$G(X_1, X_2) = \begin{pmatrix} \beta_1 \frac{SE}{N} + \beta_2 \frac{SI}{N} + \beta_3 \frac{VE}{N} + \beta_4 \frac{VI}{N} - \theta E \\ \theta E - (\gamma_1 + \gamma_2 + \gamma_3) I \end{pmatrix} \tag{22}$$

$$A = \frac{\partial G}{\partial X_2}(X^*, 0) = \begin{pmatrix} \beta_1 - \theta & \beta_2 \\ \theta & -(\gamma_1 + \gamma_2 + \gamma_3) \end{pmatrix} \tag{23}$$

A is an M-matrix since all its off-diagonal elements are non-negative.

$$G^*(X_1, X_2) = AX_2 - G(X_1, X_2) = \begin{pmatrix} (\beta_1 - \beta_1 \frac{S}{N} - \beta_3 \frac{S}{N})E + (\beta_2 - \beta_2 \frac{S}{N} - \beta_4 \frac{S}{N})I \\ 0 \end{pmatrix} \tag{24}$$

Since $0 \leq S \leq N$ and $0 \leq V \leq N$ therefore $G^*(X_1, X_2) \geq 0$. So, the two conditions (i) and (ii) are satisfied. Hence the DFE is globally asymptotically stable.

IV. SENSITIVITY ANALYSIS

The sensitivity index of each parameter that correlates with the basic reproductive numbers R_0 has been calculated and presented in table (1), and the graphical bar-graph results have been obtained in figures (2). The sensitivity analysis for this epidemic threshold illustrates the importance of each parameter to the covid-19 transmission to discover which parameter has a high impact on R_0.

The normalized forward sensitivity index of R_0 which is differentiable with respect to a given parameter ρ is defined by

$$\zeta_\rho^{R_0} = \frac{\partial R_0}{\partial \rho} \frac{\rho}{R_0} \tag{25}$$

Table 1: The sensitivity index of each parameter that correlates with R_0

CasesCases	^ R_0_	Case 1		Case 2		Case 3	
Parameters	calculation	Values	S. index	Values	S.index	Values	S. index
_1	$\frac{1}{\theta}\frac{\beta_1}{R_0}$	0.08	0.25	0.035	0.1795	0.025	0.2911
_2	$\frac{1}{(\gamma_1+\gamma_2+\gamma_3)}\frac{\beta_2}{R_0}$	0.06	0.75	0.04	0.8205	0.3	0.7089
_1	$\frac{-\beta_2}{(\gamma_1+\gamma_2+\gamma_3)^2}\frac{\gamma_1}{R_0}$	0.02	-0.4286	0.02	-0.4689	0.04	-0.4109
_2	$\frac{-\beta_2}{(\gamma_1+\gamma_2+\gamma_3)^2}\frac{\gamma_2}{R_0}$	0.01	-0.2143	0.01	-0.2344	0.02	-0.2055
_3	$\frac{-\beta_2}{(\gamma_1+\gamma_2+\gamma_3)^2}\frac{\gamma_3}{R_0}$	0.005	-0.1071	0.005	-0.1172	0.09	-0.0925
	$\frac{-\beta_1}{\theta^2}\frac{\theta}{R_0}$	0.14	-0.2500	0.14	-0.1795	0.14	-0.2911

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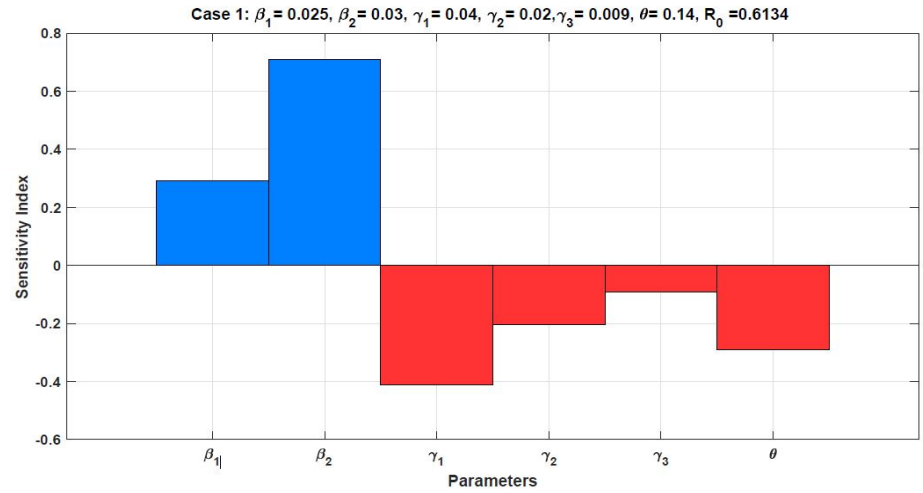
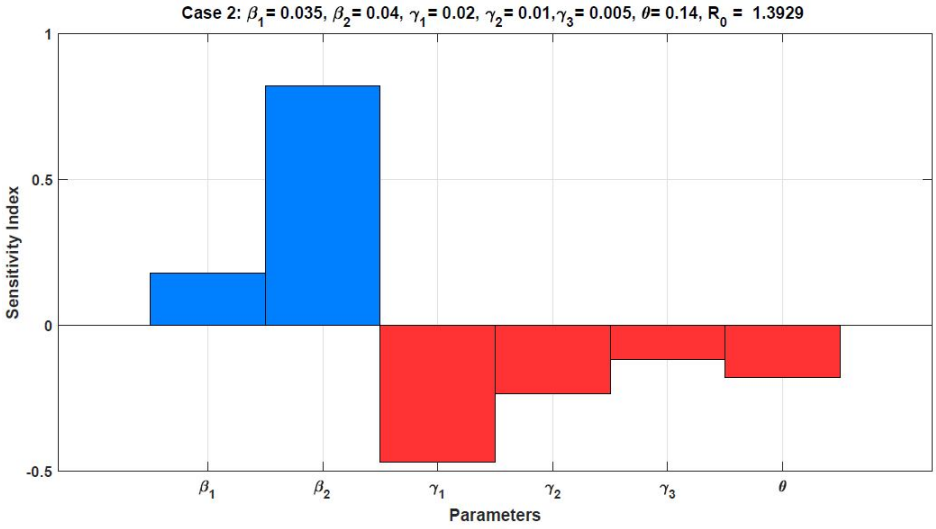
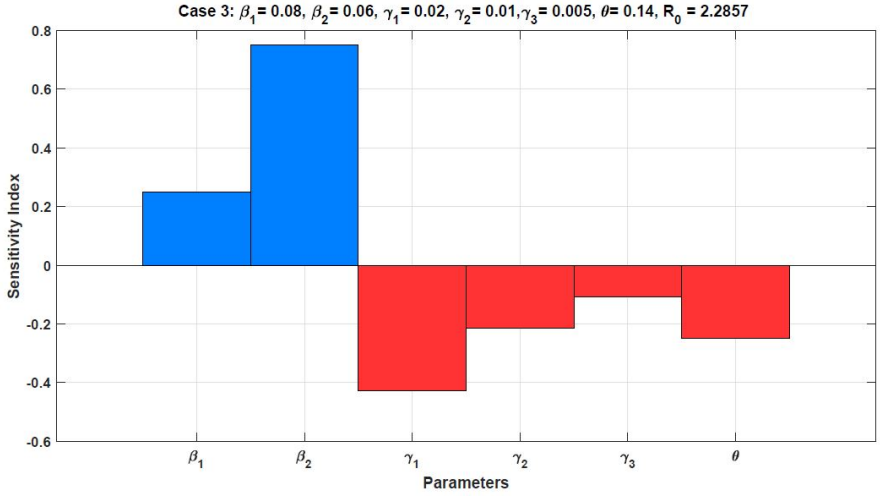


Case 1



Case 2



Case 3

Figure 2: Sensitivity analysis for Cases $1-3$ in order from top to bottom of R_0 with respect to the parameters

In general, all sensitivity cases show that some parameters are most significant to the changes in the basic reproductive number R_0. Those parameters are β_2 and γ_1. The sensitivity index illustrates that contact with asymptomatic individuals has the chance of the COVID-19 transmission. Therefore, if $\zeta_{\beta_2}^{R_0} = 0.1795$, this means if the parameter β_2 increased (or decreased) by 17.95% then R_0 will increase (or decrease) by 17.95% . The same for $\zeta_{\gamma_1}^{R_0} = -0.4689$ the decrease (or increase) of the parameter γ_1 by 46.89% will decrease (or increase) R_0 by the same percentage.

V. THE OPTIMAL ASSUMED VACCINATION

Optimal control techniques are of great use in developing optimal strategies to control various kinds of diseases. To solve the challenges of obtaining an optimal vaccination strategy, we use optimal control theory. We consider controlled system described by

$$\begin{aligned}
\frac{dS}{dt} &= -\beta_1 \frac{SE}{N} - \beta_2 \frac{SI}{N} - \lambda_1 S \\
\frac{dV}{dt} &= \lambda_1 S - \frac{V}{N}(1-u)(\beta_3 E + \beta_4 I) - \lambda_2 V \\
\frac{dE}{dt} &= \beta_1 \frac{SE}{N} + \beta_2 \frac{SI}{N} + \frac{V}{N}(1-u)(\beta_3 E + \beta_4 I) - \theta E \\
\frac{dI}{dt} &= \theta E - (\gamma_1 + \gamma_2 + \gamma_3) I \\
\frac{dQ}{dt} &= \gamma_1 I - (\sigma_1 + \delta_1) Q \\
\frac{dH}{dt} &= \gamma_2 I + \sigma_1 Q - (\sigma_2 + \delta_2) H \\
\frac{dC}{dt} &= \gamma_3 I + \sigma_2 H - (\mu + \delta_3) C \\
\frac{dR}{dt} &= \delta_1 Q + \delta_2 H + \delta_3 C + \lambda_2 V \\
\frac{dD}{dt} &= \mu C
\end{aligned} \tag{26}$$

We consider the control variable $u(t) \in U_{ad}^T$ to be the percentage of susceptible individuals being vaccinated per unit of time. Here $U_{ad}^T = \{u \mid u(t) \ \ is \ measurable\ ,\ 0 \le u(t) \le 1,\ t \in [0,T]\ \}$ indicates an admissible control set. Now, we consider an optimal control problem to minimize the objective functional

$$J(u) = \int_0^T [-A_1 V(t) + A_2 E(t) + A_3 I(t) - A_4 R(t) + \frac{1}{2}\tau u^2(t)]dt. \tag{27}$$

Here A_1, A_2, A_3 and A_4 are positive constants to keep a balance in the size of $V(t)$, $E(t)$, $I(t)$ and $R(t)$, respectively. The square of the control variable reflects the severity of the side effects of the vaccination. In the objective functional and τ is a positive weight parameter which is associated with the control $u(t)$.

In order to find an optimal solution, first we find the Lagrangian and Hamiltonian for the optimal control problem ((27)). In fact, the Lagrangian of the optimal problem is given by $L = -A_1 V(t) + A_2 E(t) + A_3 I(t) - A_4 R(t) + \frac{1}{2}\tau u^2(t)$. We seek the minimal value of the Lagrangian. To accomplish this, we define the Hamiltonian $\overline{H}$ for the control problem:

$$\begin{aligned}
\overline{H} = L + \lambda_1(t)\frac{dS(t)}{dt} + \lambda_2(t)\frac{dV(t)}{dt} + \lambda_3(t)\frac{dE(t)}{dt} + \lambda_4(t)\frac{dI(t)}{dt} + \\
\lambda_5(t)\frac{dQ(t)}{dt} + \lambda_6(t)\frac{dH(t)}{dt} + \lambda_7(t)\frac{dC(t)}{dt} + \lambda_8(t)\frac{dR(t)}{dt} + \lambda_9(t)\frac{dD(t)}{dt}
\end{aligned} \tag{28}$$

where $\zeta_1, \dots, \lambda_9$ are the adjoint functions to be determined suitably. There exists an optimal control $u^*(t)$ such that

$$J(u^*(t)) = \min_{u \in U_{ad}^T} J(u(t)).$$

Proof. To prove the existence of an optimal control we use the result in (Fleming and Rishel). Note that the control and the state variables are nonnegative values. In this minimizing problem, the necessary convexity of the objective functional in u(t) is satisfied.

The control space

$$U_{ad}^T = \{u \mid u(t) \ \ is \ measurable\ ,\ 0 \le u(t) \le 1,\ t \in [0,T]\}$$

is also convex and closed by definition. The optimal system is bounded which determines the compactness needed for the existence of the optimal control. In addition, the integrand in the functional ((27)), $A_1V(t) + A_2E(t) + A_3I(t) - A_4R(t) + \frac{1}{2}\tau u^2(t)$ is convex on the control $u(t)$. Also, we can easily see that, there exist a constant $\rho > 1$, positive numbers w_1 and w_2 such that $J(u(t)) \geq -w_2 + w_1(|v|^2)^{\frac{\rho}{2}}$. We conclude that there exists an optimal control.

A. Characterization of the optimal control

In the previous section we show the existence of an optimal control which minimize the functional ((27)). In order to derive the necessary conditions for this optimal control, we apply Pontryagin's maximum principle to the Hamiltonian H. Let $S^*(t)$, $V^*(t)$, $E^*(t)$, $I^*(t)$, $Q^*(t)$, $H^*(t)$, $C^*(t)$, $R^*(t)$, and $D^*(t)$ be optimal state solutions with associated optimal control variable $V^*(t)$ for the optimal control problem ((27)). Then, there exist adjoint variables $\zeta_1, \dots, \zeta_9$, that satisfy

$$
\begin{aligned}
\dot{\zeta}_1(t) &= (\beta_1 \tfrac{E(t)}{N} + \beta_2 \tfrac{I(t)}{N})(\zeta_1(t) - \zeta_3(t)) + \lambda_1\zeta_2(t)\\
\dot{\lambda}_2(t) &= A_1 + (1-u(t))(\beta_3 \tfrac{E(t)}{N} + \beta_4 \tfrac{I(t)}{N})(\zeta_2(t) - \zeta_3(t)) + \lambda_2(\zeta_2(t) - \zeta_8(t))\\
\dot{\zeta}_3(t) &= -A_2 + \beta_1 \tfrac{S(t)}{N}(\zeta_1(t) - \zeta_3(t)) + \beta_3(1-u(t))\tfrac{V(t)}{N}(\zeta_2(t) - \zeta_3(t)) + \theta(\zeta_3(t) - \zeta_4(t))\\
\dot{\zeta}_4(t) &= -A_3 + \beta_2 \tfrac{S(t)}{N}(\zeta_1(t) - \zeta_3(t)) + \beta_4(1-u(t))\tfrac{V(t)}{N}(\zeta_2(t) - \zeta_3(t)) + (\gamma_1+\gamma_2+\gamma_3)\zeta_4(t) - \gamma_1\zeta_5(t) - \gamma_2\zeta_6(t) - \gamma_3\zeta_7(t)\\
\dot{\zeta}_5(t) &= (\sigma_1 + \delta_1)\zeta_5(t) - \sigma_2\zeta_6(t) - \delta_1\zeta_8(t)\\
\dot{\zeta}_6(t) &= (\sigma_2 + \delta_2)\zeta_6(t) - \sigma_2\zeta_7(t) - \delta_2\zeta_8(t)\\
\dot{\zeta}_7(t) &= (\mu + \delta_3)\zeta_7(t) - \delta_3\zeta_8(t) - \mu\zeta_9(t)\\
\dot{\zeta}_8(t) &= A_4\\
\dot{\zeta}_9(t) &= 0
\end{aligned}
\tag{29}
$$

with transversality conditions

$$\lambda_i(T) = 0, \quad i = 1, \dots, 9. \tag{30}$$

Furthermore, the optimal control V^* is given by

$$u^*(t) = \max\{\min\{\tfrac{V(t)}{\tau N}(\beta_3 E(t) + \beta_4 I(t))(\zeta_3(t) - \zeta_2(t)), 1\}, 0\}. \tag{31}$$

Proof. We use Hamiltonian ((28)) in order to determine the adjoint equation and the transversality conditions. From setting $S(t) = S^*(t)$, $V(t) = E^*(t)$, $E(t) = E^*(t)$, $I(t) = I^*(t)$, $Q(t) = Q^*(t)$, $H(t) = H^*(t)$, $C(t) = C^*(t)$, $R(t) = R^*(t)$ and $D(t) = D^*(t)$, the adjoint equations and transversality conditions can be obtained by using Pontryaginâ€™s maximum principle [5, 10, 11] such that

$$
\begin{aligned}
\frac{d\zeta_1}{dt} = -\frac{d\overline{H}(t)}{dS(t)} &= -\left[\zeta_1(t)\left(-\frac{\beta_1 E(t)}{N} - \frac{\beta_2 I(t)}{N} - \lambda_1\right) + \lambda_1\zeta_2(t) + \zeta_3(t)\left(\frac{\beta_1 E(t)}{N} + \frac{\beta_2 I(t)}{N}\right)\right]\\
&= \left(\frac{\beta_1 E(t)}{N} + \frac{\beta_2 I(t)}{N}\right)(\zeta_1(t) - \zeta_3(t)) + \lambda_1\zeta_2(t)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\zeta_2}{dt} = -\frac{d\overline{H}(t)}{dV(t)} &= -\Big[-A_1 + \zeta_2(t)\left(-\beta_3(1-u(t))\frac{E(t)}{N} - \beta_4(1-u(t))\frac{I(t)}{N} - \lambda_2\right) +\\
&\quad \lambda_3\left(\beta_3(1-u(t))\frac{E(t)}{N} + \beta_4(1-u(t))\frac{I(t)}{N}\right) + \lambda_2\zeta_8(t)\Big]\\
&= A_1 + (1-u(t))(\beta_3 \tfrac{E(t)}{N} + \beta_4 \tfrac{I(t)}{N})(\zeta_2(t) - \zeta_3(t)) + \lambda_2(\zeta_2(t) - \zeta_8(t))
\end{aligned}
$$

$$\frac{d\zeta_3}{dt} = -\frac{d\overline{H}(t)}{dV(t)} = -\Big[A_2 - \beta_1\frac{S(t)}{N}\zeta_1(t) - \beta_3(1-u(t))\frac{V(t)}{N}\zeta_2(t) + \zeta_3(t)\Big(\beta_1\frac{S(t)}{N} + \beta_3(1-u(t))\frac{V(t)}{N} - \theta\Big) + \theta\zeta_4(t)$$
$$= -A_1 + \beta_1\frac{S(t)}{N})(\zeta_1(t) - \zeta_3(t)) + \beta_3(1-u(t))\frac{V(t)}{N}(\zeta_2(t) - \zeta_3(t)) + \theta(\zeta_3(t) - \zeta_4(t))$$

$$\frac{d\zeta_4}{dt} = -\frac{d\overline{H}(t)}{dV(t)} = -\Big[A_3 - \beta_2\frac{S(t)}{N}\zeta_1(t) - \beta_4(1-u(t))\frac{V(t)}{N}\zeta_2(t) + \zeta_3(t)\Big(\beta_2\frac{S(t)}{N} + \beta_4(1-u(t))\frac{V(t)}{N}\Big) + (\gamma_1+\gamma_2+\gamma_3)\zeta_4(t) + \gamma_1\zeta_5(t) + \gamma_2\zeta_6(t) + \gamma_3\zeta_7(t)$$
$$= -A_3 + \beta_2\frac{S(t)}{N})(\zeta_1(t) - \zeta_3(t)) + \beta_4(1-u(t))\frac{V(t)}{N}(\zeta_2(t) - \zeta_3(t))(\gamma_1+\gamma_2+\gamma_3)\zeta_4(t) - \gamma_1\zeta_5(t) - \gamma_2\zeta_6(t) - \gamma_3\zeta_7(t)$$

$$\frac{d\zeta_5}{dt} = -\frac{d\overline{H}(t)}{dQ(t)} = (\sigma_1+\delta_1)\zeta_5(t) - \sigma_1\zeta_6(t) - \delta_1\zeta_8$$

$$\frac{d\zeta_6}{dt} = -\frac{d\overline{H}(t)}{dH(t)} = (\sigma_2+\delta_2)\zeta_6(t) - \sigma_2\zeta_7(t) - \delta_2\zeta_8$$

$$\frac{d\zeta_7}{dt} = -\frac{d\overline{H}(t)}{dC(t)} = (\mu+\delta_3)\zeta_7(t) - \delta_3\zeta_8(t) - \mu\zeta_9$$

$$\frac{d\zeta_8}{dt} = -\frac{d\overline{H}(t)}{dR(t)} = A_4$$

$$\frac{d\zeta_9}{dt} = -\frac{d\overline{H}(t)}{dD(t)} = 0$$

And by using the optimality conditions we find

$$\frac{d\overline{H}(t)}{du(t)} = \tau u^*(t) + \beta_3\frac{V(t)E(t)}{N}\zeta_2(t) + \beta_4\frac{V(t)I(t)}{N}\zeta_2(t) - \beta_3\frac{V(t)E(t)}{N}\zeta_3(t) - \beta_4\frac{V(t)I(t)}{N}\zeta_3(t) = 0,$$

which gives

$$u^*(t) = \frac{V(t)}{\tau N}(\beta_3E(t) + \beta_4I(t))(\zeta_3(t) - \zeta_2(t)).$$

Using the property of the control space, we obtain

$$\begin{cases} u^* = 0 & if \quad \frac{V(t)}{\tau N}(\beta_3E(t) + \beta_4I(t))(\zeta_3(t) - \zeta_2(t)) \le 0 \\ u^* = \frac{\mu N(\lambda_1-\lambda_4)}{\tau} & if \quad 0 < \frac{V(t)}{\tau N}(\beta_3E(t) + \beta_4I(t))(\zeta_3(t) - \zeta_2(t)) < 1 \\ u^* = 1 & if \quad \frac{V(t)}{\tau N}(\beta_3E(t) + \beta_4I(t))(\zeta_3(t) - \zeta_2(t)) \ge 1. \end{cases}$$

So the optimal control is characterized as

$$u^* = \max\{\min\{\frac{V(t)}{\tau N}(\beta_3E(t) + \beta_4I(t))(\zeta_3(t) - \zeta_2(t)),1\},0\}. \quad (32)$$

VI. NUMERICAL SIMULATION

The data in Covid-19 was collected from the World Health Organization [8] and Ministry of Health in Kenya [4]. The estimated total population is $N \simeq 44 \times 10^6$ and the initial conditions depended on Covid-19 data registered between March and October 2020 and these are listed. The fitting of data of the total infected persons for the period under study

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was done by estimating the values of the parameters as follows: For $\beta_1 = 0.14$, $\beta_2 = 0.16$, $\beta_3 = 0.04$, $\beta_4 = 0.06$, $\lambda_1 = 0.03$, $\lambda_2 = 0.01$, $\theta = 0.1$, $\gamma_1 = 0.02$, $\gamma_2 = 0.01$, $\gamma 3 = 0.005$, $\sigma_1 = 0.003$, $\sigma_2 = 0.001$, $\delta_1 = 0.001$, $\delta_2 = 0.002$;, $\delta_3 = 0.002$, $\mu = 0.01$, $S_0 = 2000$, $V_0 = 20$, $E_0 = 240$, $I_0 = 1$, $Q_0 = 50$, $H_0 = 15$, $C_0 = 14$, $R_0 = 0$ and $D_0 = 0$.

Figure 3 shows the simulation of the number of infected individuals in Kenya between March and October 2020. The R^2 shows the variability of the data from the assumed real figures. The $R^2 = 99\% \simeq 100$, indicating there is hardly any variability of the simulated data and real data.

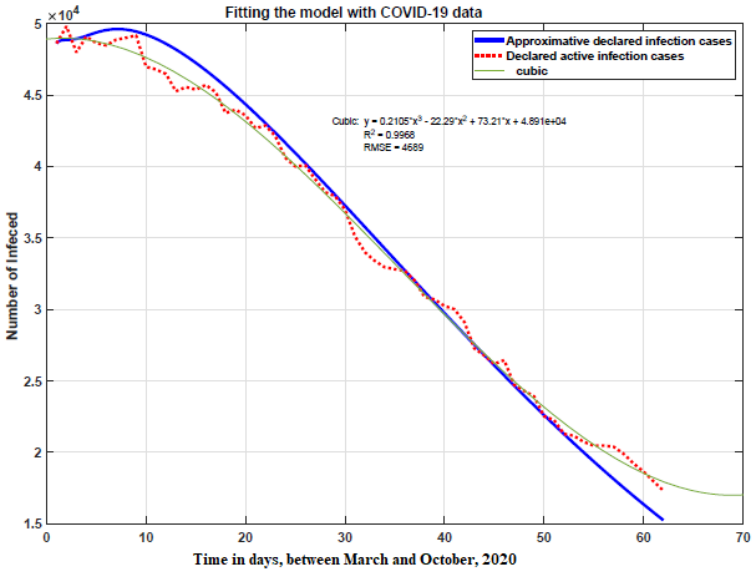

Figure 3: Evolutionary dynamics of the model 1

Figure 4 show the simulation of individual in compartments based on the model equations (1).

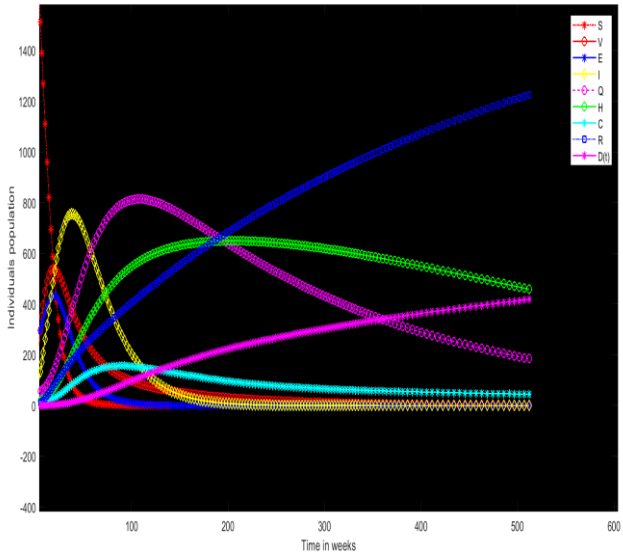

Figure 4: Evolutionary dynamics of the model 1

Figure 4 indicate that at the onset of the pandemic, few persons are infected. The number of infected individuals grows exponentially as the infection period increases. Figure 4 reveals that the epidemic would naturally die after approximately 512 weeks as the number of infected individuals would be 0 as the recovered hits the maximum number. Figure 5 shows the simulated susceptible, infectious, and recovered individuals under control (vaccination) or no vaccination.

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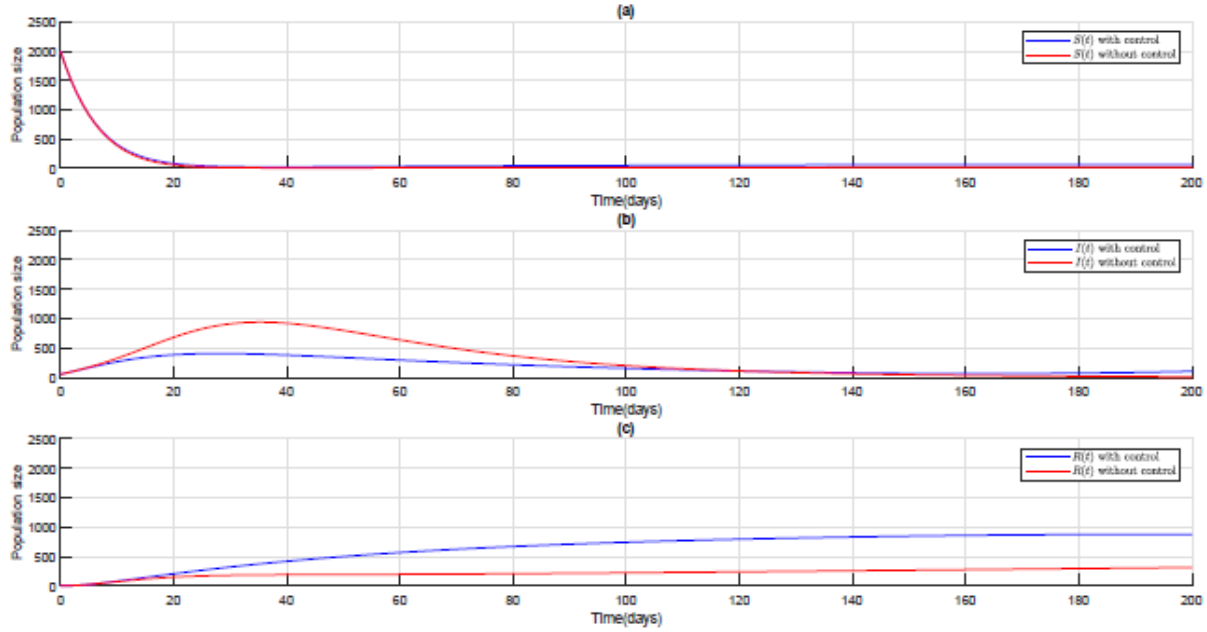

Figure 5: Evolutionary dynamics of the model for susceptible, infectious and recovered individuals under control (vaccination) or without control (no vaccination).

Figure 5a shows the susceptible population shows no difference between the two conditions, that is under vaccination or no vaccination. Figure 5b shows that for the infected population, without vaccination, the number of those infected would reach the peak in approximately 45 days then subside to fit the trend line of hose with vaccination. Figure 5c shows that for the recovered population, without vaccination, the number of those recovered would always be higher than those without control. These observations suggest that vaccination would be effective in controlling the epidemic as the number of infected would be lower and recovered higher. Figure 6 simulate the number of days after which the vaccination would be effective after application.

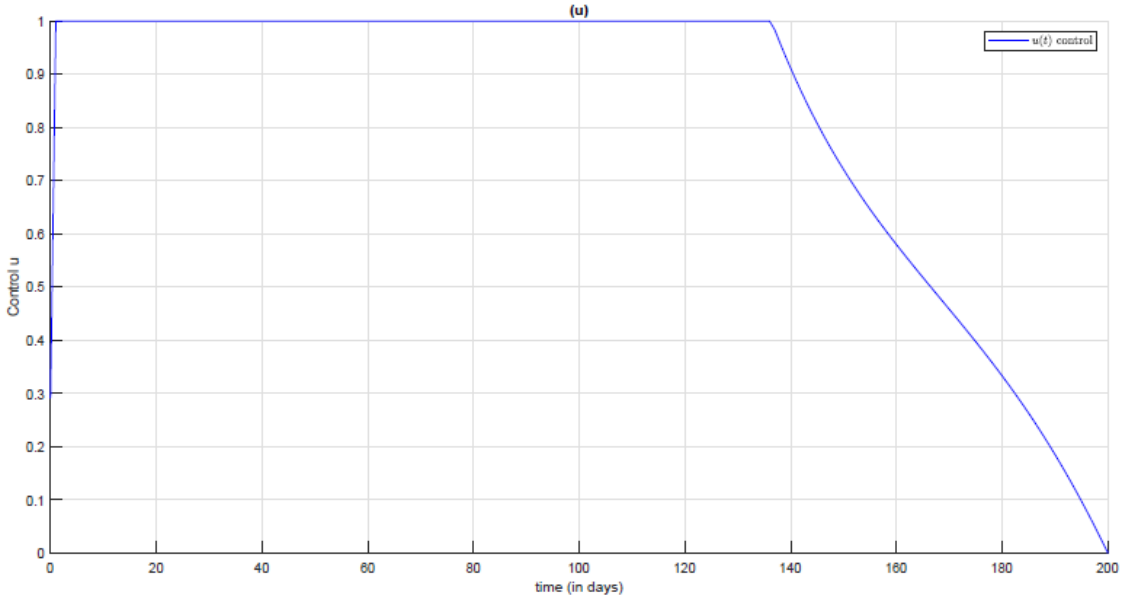

Figure 6: The number of days upon which the vaccination or optimal control (32) would be effective

Figure 6 suggests that the optimal control, which is vaccination would be effective after ≈ 138 days. This suggests that the vaccination would begin to control the spread of Covid-19 and by 200 days, the epidemic would have stopped.

VII. CONCLUSION

The study has presented a mathematical model of Covid-19. The basic reproductive number of the model have been obtain. The local and global stability analysis of the model have been established. The sensitivity analysis of the model indicate per capita density infection rate are the most sensitive. The vacination as the optimal control has been presented and characterised. The study via mathematical and numerical simulations based on Kenyan data has shown the effectiveness of an assumed vaccination. The model vaccination is show to be effective in controlling the infection rate. The simulated graphs suggest that the model vaccination would begin to take effect on the pandemic after 138 days and stop the epidemic after 200 days. This study, though not perfect since it is based on assumed and estimated parameters, would be informative to the medical stakeholders. Future studies would incorporate real parameter values to present the exact scenarios for accurate estimation.

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